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This proposal describes work to be carried out extending the study of phase-space density representations of fluid dynamics to the case where the fluid density $p(\vec{x},t)$ is not constant, but the fluid is incompressible, so the fluid velocity $\vec{u}(\vec{x},t)$satisfies $\nabla\cdot\vec{u}=0$. This will allow the phase-space density representation to describe inhomogeneous fluids. Since the phase space density representation is ideally suited to making time scale separations of the form needed for isolating geostrophic and quasi-geostrophic motions of the full fluid equations and doing so in a fully Hamiltonian fashion, this improvement on past work will enable the inclusion of density variations into the Hamiltonian formulation of these mesoscale geophysical motions.

The extension of the phase space density formulation of fluids to this case will allow the study of the interplay between internal wave motions-fast density variations-and mesoscale motions-far slower geostrophic and quasi-geostrophic motions.

The proposal describes, in outline, the work which will be carried out by the principal investigator and a graduate student over a period of three years which will formulate the Hamiltonian version of inhomogeneous, incompressible flows in phase space density form and separate out the slow and fast motions relevant to geophysical flows. The equations so derived will be studied numerically both as a Hamiltonian system and with the addition of realistic models of forcing and damping. One of the goals of the study is to investigate the interplay between the slow geostrophic and near geostrophic modes and their transfer of energy to the much faster inertial and internal wave modes. This process would appear to be damping on the meoscale flow which would determine their lifetime in terms of parameters connected with internal wave motions.

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Hamiltonian Dynamics of Coupled Potential Vorticity and Internal Wave Motion: I. Linear Modes

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I. Introduction

The study of stratified fluid dynamics has primarily concentrated on the linear or nonlinear interaction of familiar modes of density oscillations in the internal wave frequency range between the local inertial frequency, f, and the local buoyancy frequency, N. It has been observed by Müller and his coworkers, Müller (1988) and Müller et al. (1986,1988), that in the linearization of the Eulerian equations for internal wave motion, in addition to the familiar internal wave oscillations, there is also a zero frequency mode which is clearly identifiable as geostrophic motion or more precisely motion which possesses potential vorticity and whose space and time scales are to be established. In these papers the question of whether there is observational or computational evidence, Riley et al. (1981), for these potential vorticity carrying modes has been addressed at some length. The evidence is quite strong that these modes do exist, and in some sense of equipartition carry as much energy as the familiar, zero potential vorticity, internal wave motions.

The purpose of this paper is to provide the consistent formulation of the dynamics of these modes by studying their interaction in some detail in a fully Hamiltonian context using a Lagrangian specification of the fluid dynamics of a stratified fluid. The reasons for using the Lagrangian view of the fluid dynamics are twofold: (1) the formulation is canonical and all the familiar tools of Hamiltonian mechanics can be employed without worry about the noncanonical Poisson brackets which arise in the Eulerian fluid dynamics. (2) The conserved quantities which appear as cyclic coordinates in a canonical formulation (in particular the potential vorticity) have been "reduced" out of the Eulerian dynamics by construction, and thus they appear as mysterious or at least obscure consequences of the Eulerian equations of motion. As cyclic coordinates in the Lagrangian formulation, one may treat them on an equal footing with the interacting internal wave modes. Once this is done, the equipartition of energy among these modes appears as a familiar consequence of similar arguments for equi-distribution of energy among modes in general Hamiltonian systems.

A word about "reduction" is in order here, (Abarbanel et al. (1986)). This is the process whereby one goes from the six fields in a Lagrangian picture: the Lagrangian positions and their conjugate momenta, to the four fields in an Eulerian framework: the three fluid velocities and the fluid density. The remaining quantities are constants of the motion, and the reduced description provided by the Eulerian formalism restricts evolution of the fluid to surfaces where these quantities are constant. Thus the Eulerian formulation is unable to shed any dynamical light on the evolution of these degrees of freedom and leaves us in the dark about how the degrees of freedom in, for example, potential vorticity interact with the internal wave modes. The Lagrangian formulation which keeps all degrees of freedom overcomes these complications.

Having identified the modes which carry potential vorticity, we associate them with both small scale oscillations which would contribute to carrying energy in the internal wave frequency range, $f \leq \omega \leq N$ and with the large scale mesoscale motions which drive the internal waves by transfers of energy to them, Müller (1976,1988), Müller et al. (1986,1988) and Brown and Evans (1981). [We refer to these modes as PV modes to indicate that they carry potential vorticity regardless of their spatial scales.] This formulation then allows us to investigate the interesting question of the transfers of energy and momentum from mesoscale geostrophic modes to the internal wave degrees of freedom using familiar Hamiltonian mechanics. The problem is certainly harder to formulate in Eulerian terms. If the initial conditions of the problem were chosen to have the potential vorticity modes concentrated at small scales only, then the interaction among the modes would be essentially a small scale energy transfer issue, though eventually the energy would affect the larger scale motions as a matter of principle. The flow from the larger scales to the smaller scale internal wave and 'vortical' modes presumably occurs on a much more rapid time scale than energy flow in the other direction, if indeed, the latter can occur at all before being dissipated at the smallest scales by viscosity.

We also formulate the problem in terms of a *compressible* fluid and would recover the incompressible case of more direct physical interest by taking the physical limit of sound speed going to infinity at the end of any calculations. This requires us to carry acoustic modes of the fluid motion along with internal waves modes, but it also allows us to ignore the constraint of incompressibility throughout the work. Since this constraint is peculiar and difficult to implement in the Hamiltonian framework, it is convenient to have it out of the way. Lighthill (1978) shows in a linear context that as the sound speed becomes large compared to fluid motion speeds, the acoustic and other modes decouple, as one would certainly expect on physical grounds.

Our work proceeds from the general Lagrangian formulation of the fluid dynamics for three dimensional flows. We linearize the canonical degrees of freedom around a base state of no flow and identify the quadratic terms in the Hamiltonian for the problem. After a discussion of the dispersion relation of the linearized problem, we exhibit a canonical transformation from the original canonical coordinates into a set which explicitly possesses the conserved potential vorticity as one of the canonical momenta whose canonical coordinate is absent from the Hamiltonian. That is, potential vorticity (PV) appears as a cyclic coordinate in the quadratic Hamiltonian. Next, we impose periodic boundary conditions, and identify appropriately scaled Fourier coefficients as canonical variables. It is then possible to explicitly decouple the vortical, internal wave and acoustic modes by a further canonical transformation. In other words the quadratic Hamiltonian is expressed as a sum of independent harmonic oscillator Hamiltonians. Finally we formulate the problem of interaction among the modes identified in the linear problem and end with a discussion of future work including both numerical directions and a possible way to utilize the full, *nonlinear* potential vorticity as a cyclic canonical momentum in a further canonical set of variables for the problem. This paper treats the linear modes only. Our work on the nonlinear interaction among the modes identified here will appear in future publications. In our final section we provide both a summary of the work in the paper and some suggestions on the work to be done in the nonlinear problem. In particular we note that the potential vorticity in Lagrangian formulation is quadratic in the modes we identify in the linear problem. This is in contrast to the Eulerian formulation where the potential vorticity is an infinite series in the velocity and density perturbations to the base state. This suggests that working to all orders in the nonlinear problem may well be simpler in Lagrangian picture especially if one wants to emphasize the matter of potential vorticity conservation.

II. Action Principle in Lagrangian Formulation

Our starting point is the Lagrangian for the fluid dynamics of a compressible fluid with an energy density $\epsilon(\rho)$ and canonical coordinate $\mathbf{Y}(\mathbf{r},t)$, which is the Lagrangian position vector of a fluid particle with label $\mathbf{r}$. This label is also the initial position of the fluid particle for we require $\mathbf{Y}(\mathbf{r},0)=\mathbf{r}$. We want the motion to take place in a rotating frame, so we add the "rotational potential" $\mathbf{R}(\mathbf{Y},t)$ whose curl with respect to $\mathbf{Y}$ is twice the local rotation of the earth. In this paper we will work entirely in an f-plane framework, thus we can take

$$\mathbf{R}(\mathbf{Y},t)=\frac{f}{2}(-\hat{\mathbf{x}}Y_2+\hat{\mathbf{y}}Y_1) \tag{1}$$

for rotation about the z-axis at $f/2$.

For this model we have the stationary action principle, Abarbanel and Holm (1987), for

$$\int_{t_1}^{t_2} dt \int d^3r \, \rho_0(\mathbf{r}) \left[\frac{1}{2} \frac{\partial \mathbf{Y}(\mathbf{r},t)}{\partial t}^2 + \frac{\partial \mathbf{Y}(\mathbf{r},t)}{\partial t} \cdot \mathbf{R}(\mathbf{Y}(\mathbf{r},t)) - \Phi(\mathbf{Y}(\mathbf{r},t)) - \epsilon(\rho) \right], \tag{2}$$

where $\rho_0(\mathbf{r})$ is the density at time $t = 0$, $\Phi(\mathbf{Y}) = gY_3$, and g is the gravitational acceleration. The density ρ is

$$\rho(\mathbf{Y}(\mathbf{r},t),t) = \rho_0(\mathbf{r}) \frac{\partial(\mathbf{r})}{\partial(\mathbf{Y}(\mathbf{r},t))}, \tag{3}$$

and is a derived quantity from the dynamical variables $\mathbf{Y}(\mathbf{r},t)$ and $\partial \mathbf{Y}(\mathbf{r},t)/\partial t$ with the time derivatives taken at fixed $\mathbf{r}$.

Varying this action with respect to $\mathbf{Y}(\mathbf{r},t)$ with variations vanishing at the times t_1 and t_2 results in the equations of motion

$$\frac{\partial^2 \mathbf{Y}(\mathbf{r},t)}{\partial t^2} = -\frac{1}{\rho(\mathbf{Y}(\mathbf{r},t))} \nabla_Y p(\mathbf{Y}(r,t)) - \nabla_Y \Phi(\mathbf{Y}(\mathbf{r},t)) + \frac{\partial \mathbf{Y}(\mathbf{r},t)}{\partial t} \times \hat{\mathbf{z}} f, \tag{4}$$

in which $p(\mathbf{Y}(\mathbf{r},t)) = \rho^2 \epsilon_\rho(\rho)$.

The Hamiltonian is established in the usual fashion. Calling the integrand in Eq.(2) the Lagrangian $L(\mathbf{Y}(\mathbf{r},t), \partial_t \mathbf{Y}(\mathbf{r},t))$ we have the canonical momentum

$$\boldsymbol{\Pi}(\mathbf{r},t) = \frac{\partial L}{\partial[\partial_t \mathbf{Y}(\mathbf{r},t)]}, \tag{5}$$

or

$$\boldsymbol{\Pi}(\mathbf{r},t) = \rho_0(\mathbf{r})[\partial_t \mathbf{Y}(\mathbf{r},t) + \mathbf{R}(\mathbf{Y},t)], \tag{6}$$

and this leads to the familiar Hamiltonian

$$H(\mathbf{Y},\boldsymbol{\Pi}) = \int d^3r \left[\frac{|\boldsymbol{\Pi} - \rho_0 \mathbf{R}(\mathbf{Y})|^2}{2\rho_0} + \rho_0 \Phi(\mathbf{Y}) + \rho_0 \epsilon(\rho) \right]. \tag{7}$$

III. Hamiltonian for the Linearized Motion

We want to isolate the linear terms in the equations of motion which arise from this Hamiltonian and the canonical Poisson brackets

$$\{Y_i(\mathbf{r},t), \Pi_j(\mathbf{r}',t)\} = \delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}'); \tag{8}$$

$i,j = 1,2,3$. For this purpose we want the quadratic terms in the Hamiltonian in the expansion of the dynamical variables around the base state $\mathbf{Y}(\mathbf{r},t) = \mathbf{r}$ and $\partial_t\mathbf{Y}(\mathbf{r},t) = 0$ using

$$\mathbf{Y}(\mathbf{r},t) = \mathbf{r} + \mathbf{X}(\mathbf{r},t), \tag{9}$$

and

$$\boldsymbol{\Pi}(\mathbf{r},t) = \frac{\rho_0 f}{2}\hat{\mathbf{z}} \times \mathbf{r} + \boldsymbol{\pi}(\mathbf{r},t). \tag{10}$$

The only tricky part of the expansion procedure is in capturing the quadratic terms in the energy density $\epsilon(\rho) = \epsilon(\rho_0\partial(\mathbf{r})/\partial(\mathbf{Y}))$. For this we need the result that when $\mathbf{y} = \mathbf{r} + \mathbf{X}(\mathbf{r})$

$$\frac{\partial(\mathbf{Y})}{\partial(\mathbf{r})} = 1 + \nabla\cdot\mathbf{X} + \frac{\partial(X_1,X_2)}{\partial(r_1,r_2)} + \frac{\partial(X_1,X_3)}{\partial(r_1,r_3)} + \frac{\partial(X_2,X_3)}{\partial(r_2,r_3)} + \frac{\partial(\mathbf{X})}{\partial(\mathbf{r})}. \tag{11}$$

This leads to the Hamiltonian correct to second order in the coordinates and to all orders in π

$$H_2(\mathbf{X},\boldsymbol{\pi}) = \int d^3r \left[\frac{|\boldsymbol{\pi}|^2}{2\rho_0} + \frac{f}{2}[X_2\pi_1 - X_1\pi_2] + \frac{\rho_0 f^2}{8}X_\perp^2 + \frac{\rho_0 c^2}{2}(\nabla\cdot\mathbf{X})^2 - \rho_0 g X_3 \nabla_\perp \cdot X_\perp\right], \tag{12}$$

where the two vector $\mathbf{X}_\perp = (X_1, X_2)$, and the speed of sound is defined by $p_\rho = c^2$ evaluated at the base state $\rho = \rho_0(\mathbf{r})$.

We may rewrite this form for the quadratic Hamiltonian H_2 in the following suggestive fashion:

$$H_2(\mathbf{X},\boldsymbol{\pi}) = \int d^3r \left[\frac{|\boldsymbol{\pi}|^2}{2\rho_0} + \frac{f}{2}(X_2\pi_1 - X_1\pi_2) + \frac{\rho_0 f^2}{8}X_\perp^2 + \frac{1}{2\rho_0 c^2}(\Delta p)^2 + \frac{1}{2}\rho_0 N^2 X_3^2\right], \tag{13}$$

where the buoyancy frequency N is

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial r_3} - \frac{g^2}{c^2}, \tag{14}$$

and the acoustic pressure is

$$\Delta p = -\rho_0 c^2 \nabla \cdot \mathbf{X} + g\rho_0 X_3. \tag{15}$$

This form of the Hamiltonian shows clearly the separate contributions of acoustic energy and internal wave energy, see Lighthill (1978).

The equations of motion following from this Hamiltonian and the Poisson brackets $\{X_i(\mathbf{r}), \pi_j(\mathbf{r}')\} = \delta_{ij}\delta^3(\mathbf{r} - \mathbf{r}')$ are

$$\partial_{tt}\mathbf{X}_\perp = f\partial_t \mathbf{X}_\perp \times \hat{\mathbf{z}} - \nabla_\perp \zeta, \tag{16}$$

and

$$\partial_{tt} X_3 = -\frac{N^2 c^2}{g}(\nabla \cdot \mathbf{X}) - \partial_3 \zeta, \tag{17}$$

where $\zeta = \Delta p / \rho_0$.

It is immediate from the taking the curl of the $\mathbf{X}_\perp$ equation that the conservation law

$$\partial_t \left[\partial_1 \dot{X}_2 - \partial_2 \dot{X}_1 + f\nabla_\perp \cdot \mathbf{X}_\perp\right] = 0 \tag{18}$$

holds. This is the linearized form of the conservation law found in Abarbanel and Holm (1987) for pressure relationships of the form we have here; namely, pressure a function of ρ and, through $\rho_0(r_3)$, also a function of r_3. That conservation law states that the following quantity is conserved:

$$\Omega(\mathbf{r},\cdot \; - \sum_{n=1}^{3} \left\{ \frac{\partial Y_n}{\partial r_1}\frac{\partial}{\partial r_2}\left[\frac{\partial Y_n}{\partial t} + R_n(\mathbf{Y},t)\right] - \frac{\partial Y_n}{\partial r_2}\frac{\partial}{\partial r_1}\left[\frac{\partial Y_n}{\partial t} + R_n(\mathbf{Y},t)\right]\right\}, \tag{19}$$

so

$$\frac{\partial\Omega(\mathbf{r},t)}{\partial t} = 0. \tag{20}$$

This is the potential vorticity as shown in the paper of Abarbanel and Holm (1987).

If one considers the buoyancy frequency constant, then these linearized equations admit a plane wave solution of the form

$$\mathbf{X}(\mathbf{r},t) = \mathbf{X}^0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \tag{21}$$

with $\mathbf{X}^0$ a constant vector. The frequency is related to the wave vector by the dispersion relation

$$(\omega^2 - f^2)(\omega^2 - ik_3(g + \frac{N^2c^2}{g})) - \omega^2|\mathbf{k}|^2c^2 + c^2(k_3^2f^2 + |k_\perp|^2N^2) = 0. \tag{22}$$

This has the approximate solutions, for large c, of an acoustic mode

$$\omega_A = |\mathbf{k}|c, \tag{23}$$

and of internal wave modes

$$\omega_{IW}^2 = \frac{(k_3^2f^2 + |k_\perp|^2N^2)}{|\mathbf{k}|^2}. \tag{24}$$

There is also a zero frequency mode associated with the conservation law, and in this mode we have the condition on the dynamical variables

$$f\partial_t\mathbf{X}_\perp \times \hat{\mathbf{z}} = \nabla_\perp\zeta, \tag{25}$$

which is recognized as the geostrophic relation.

One of our goals will be to bring out the conservation law as resulting from a cyclic or ignorable coordinate in the Hamiltonian by making an appropriate canonical transformation.

The canonical transformation which isolates the conserved quantity as a canonical momentum with an ignorable coordinate is the following

$$Q_1(\mathbf{r},t) = \partial_1 X_2 - \partial_2 X_1, \tag{26}$$

$$Q_2(\mathbf{r},t) = \nabla_\perp \cdot \mathbf{X}_\perp, \tag{27}$$

$$Q_3(\mathbf{r},t) = X_3, \tag{28}$$

$$P_1(\mathbf{r},t) = \frac{-1}{\nabla_\perp{}^2}(\partial_1 \pi_2 - \partial_2 \pi_1 + \frac{\rho_0 f}{2}[\nabla_\perp \cdot \mathbf{X}_\perp]), \tag{29}$$

$$P_2(\mathbf{r},t) = \frac{-1}{\nabla_\perp{}^2}(\nabla_\perp \cdot \pi_\perp + \frac{\rho_0 f}{2}[\partial_1 X_2 - \partial_2 X_1]), \tag{30}$$

$$P_3(\mathbf{r},t) = \pi_3. \tag{31}$$

Using this transformation we find the new Hamiltonian

$$\begin{aligned} H_2(\mathbf{Q},\mathbf{P}) &= \int d^3r \left[\frac{1}{2\rho_0}\{|\nabla_\perp P_1|^2 + |\nabla_\perp P_2|^2 + P_3^2\} - fQ_2P_1 \right. \\ &+ \left. \frac{\rho_0 f^2}{2} Q_2 \frac{-1}{\nabla_\perp{}^2} Q_2 + \frac{\rho_0 c^2}{2}(Q_2 + \partial_3 Q_3)^2 - \rho_0 g Q_2 Q_3\right]. \end{aligned} \tag{32}$$

This is clearly independent of Q_1, so the canonical momentum P_1 is conserved $\partial_t P_1(\mathbf{r},t) = 0$. The coupling between P_1 and Q_2 acts as a constant displacement on the Q_2 motion. We could remove it by a noncanonical transformation, but it is harmless as it stands, so we leave it present. P_1 is directly related to the conserved quantity one deduces from the equations of motion since

$$P_1(\mathbf{r},t) = \rho_0 \frac{-1}{\nabla_\perp{}^2}\left[\partial_1 \dot{X}_2 - \partial_2 \dot{X}_1 + f\nabla_\perp \cdot \mathbf{X}_\perp\right]. \tag{33}$$

IV. A Modified Dispersion Relation

We now exhibit a further canonical transformation that explicitly decouples the internal wave, acoustic and potential vorticity (PV) motions in the Hamiltonian for the linearized motion. This provides a physically motivated set of coordinates in which to study the effects of the nonlinear interactions. We have already isolated the PV degrees of freedom from the other degrees of freedom by a canonical transformation given in the previous section. We shall see that a subtlety is involved in the use of this transformation. This will be discussed below and later another canonical transformation will be given that explicitly decouples the PV and internal wave modes for the linearized motion.

In order to accomplish this transformation to normal mode coordinates it is necessary first to make an intermediate change of variables. This is related to a peculiarity of the dispersion relation (22), which the reader may have noticed, namely the occurrence of *imaginary* quantities. This might suggest an instability of the steady state, but as hinted at in Lighthill (1978), it is the result of not looking at the problem in quite the right coordinates. A transformation to a set of proper coordinates will give a well behaved dispersion relation, with four *real* frequencies at each choice of wave number. These correspond to pairs of acoustic and internal waves travelling in opposite directions. The form of the Hamiltonian in Eq. (12) itself suggests the correct transformation by the following reasoning. One would like to make the problem of finding normal mode coordinates algebraic by using a Fourier representation of the degrees of freedom. The appearance of factors of $\rho_0(\mathbf{r})$ prevent a straightforward implementation of this plan. These factors go away by using the following canonical transformation

$$\mathbf{v}(\mathbf{r},t) = {\rho_0}^{-1/2}(\mathbf{r})\pi(\mathbf{r},t), \quad \mathbf{x}(\mathbf{r},t) = {\rho_0}^{1/2}(\mathbf{r})\mathbf{X}(\mathbf{r},t). \tag{34}$$

In terms of these variables the quadratic Hamiltonian is expressed as

$$\begin{aligned} H_2(\mathbf{x},\mathbf{v}) &= \int d^3r \left[\frac{1}{2}v^2 + \frac{f}{2}(x_2v_1 - x_1v_2) + \frac{f^2}{8}x_\perp^2 \right. \\ &+ \left. \frac{c^2}{2}(\nabla\cdot\mathbf{x})^2 + (\frac{\alpha c^2}{2} - g)x_3\nabla_\perp\cdot\mathbf{x}_\perp + \frac{\alpha^2c^2}{8}x_3^2\right], \end{aligned} \tag{35}$$

where

$$\alpha \equiv \frac{N^2}{g} + \frac{g}{c^2} = -\frac{d \log \rho_0(r_3)}{dr_3}. \tag{36}$$

Now the quadratic Hamiltonian contains no explicit ρ_0 factors so that a transformation to Fourier modes, assuming N to be constant, will indeed make the normal mode problem algebraic. Before we go on to this, it is interesting to compute the dispersion relation again, using Hamilton's equations for $\mathbf{x}$ and $\mathbf{v}$. We easily deduce

$$\partial_{tt}\mathbf{x}_\perp = f\partial_t\mathbf{x}_\perp \times \hat{\mathbf{z}} + \nabla_\perp \left[c^2 \nabla \cdot \mathbf{x} + (\frac{\alpha c^2}{2} - g)x_3 \right], \tag{37}$$

$$\begin{aligned} \partial_{tt}x_3 &= c^2\partial_3(\nabla \cdot \mathbf{x}) - \frac{\alpha^2 c^2}{4}x_3 - (\frac{\alpha c^2}{2} - g)\nabla_\perp \cdot \mathbf{x}_\perp, \\ &= -\frac{\alpha^2 c^2}{4}x_3 - (\frac{\alpha c^2}{2} - g)\nabla \cdot \mathbf{x} + \partial_3 \left[c^2 \nabla \cdot \mathbf{x} + (\frac{\alpha c^2}{2} - g)x_3 \right]. \end{aligned} \tag{38}$$

This yields the dispersion relation for plane waves

$$(\omega^2 - f^2)(\omega^2 - \frac{\alpha^2 c^2}{4})) - \omega^2 |\mathbf{k}|^2 c^2 + c^2(k_3^2 f^2 + |k_\perp|^2 N^2) = 0, \tag{39}$$

where, as before, there is also a zero frequency solution corresponding to the geostrophic motion. Comparing this dispersion relation for combined acoustic and internal wave modes to the previous dispersion relation, Eq. (22), we see that the following modification has taken place

$$ik_3 \longrightarrow \frac{\alpha}{4}. \tag{40}$$

We shall see later that the dispersion relation in its new form has four real roots.

V. Normal Mode Coordinates

We now take our fluid to be confined to a periodic box with horizontal sides of length L and height D. Our coordinates and momenta can now be expressed as Fourier series

$$x_i(\mathbf{r}, t) = \frac{1}{\sqrt{DL^2}} \sum_{\mathbf{n}} q_i(\mathbf{n}, t) \exp(i(\lambda \mathbf{n}_\perp \cdot \mathbf{r}_\perp + \kappa n_3 r_3)), \tag{41}$$

$$v_i(\mathbf{r},t) = \frac{1}{\sqrt{DL^2}}\sum_{\mathbf{n}} p_i(\mathbf{n},t)\exp(-i(\lambda\mathbf{n}_\perp\cdot\mathbf{r}_\perp + \kappa n_3 r_3)), \tag{42}$$

where

$$\lambda = \frac{2\pi}{L}, \quad \kappa = \frac{2\pi}{D}, \tag{43}$$

$$\mathbf{n} = (n_1, n_2, n_3), \quad \mathbf{n}_\perp = (n_1, n_2, 0), \tag{44}$$

and where the sum above runs over all integer values of n_1, n_2 and n_3. One must also have $q_i(-\mathbf{n},t) = q_i^*(\mathbf{n},t)$ and $p_i(-\mathbf{n},t) = p_i^*(\mathbf{n},t)$, by the reality of $x_i(\mathbf{r},t)$ and $v_i(\mathbf{r},t)$. The $\sqrt{DL^2}$ normalization assures that the q_i and p_i are canonical coordinates with Poisson bracket

$$\{q_i(\mathbf{n},t), p_j(\mathbf{m},t)\} = \delta_{ij}\delta_{\mathbf{nm}}. \tag{45}$$

Expressing the quadratic Hamiltonian, $H_2(\mathbf{x},\mathbf{v})$ Eq. (35), in terms of these Fourier coefficients, we obtain

$$\begin{aligned} H_2(q,p) &= \frac{1}{2}\sum_{\mathbf{n}}\Big[p(\mathbf{n})\cdot p(-\mathbf{n}) + f(q_2(\mathbf{n})p_1(\mathbf{n}) - q_1(\mathbf{n})p_2(\mathbf{n})) \\ &+ \frac{f^2}{4} q_\perp(\mathbf{n})\cdot q_\perp(-\mathbf{n}) + c^2(\lambda\mathbf{n}_\perp\cdot q_\perp(\mathbf{n}) + \kappa n_3 q_3(\mathbf{n}))(\lambda\mathbf{n}_\perp\cdot q_\perp(-\mathbf{n}) + \kappa n_3 q_3(-\mathbf{n})) \\ &+ i\lambda(\frac{\alpha c^2}{2} - g)(\mathbf{n}_\perp\cdot q_\perp(\mathbf{n})q_3(-\mathbf{n}) - \mathbf{n}_\perp\cdot q_\perp(-\mathbf{n})q_3(\mathbf{n})) + \frac{\alpha^2 c^2}{4} q_3(\mathbf{n})q_3(-\mathbf{n})\Big]. \end{aligned} \tag{46}$$

The evolution equations for the Fourier coefficients are Hamilton's equations

$$\frac{dq_i(\mathbf{n})}{dt} = \frac{\partial H_2}{\partial p_i(\mathbf{n})}, \quad \frac{dp_i(\mathbf{n})}{dt} = -\frac{\partial H_2}{\partial q_i(\mathbf{n})}. \tag{47}$$

It would now seem that in our search for normal mode coordinates we should first implement a Fourier version of the canonical transformation in Eq.'s (26)-(31), which isolates the PV motion. However there is an issue here which we mentioned above. An examination of the canonical transformation reveals that it is not well defined for phase space variables independent of $\mathbf{r}_\perp$ because of the occurrence of the inverse perpendicular Laplacian in the transformation. This is a warning that in the Hamiltonian (46) we should treat the case of mode amplitudes satisfying $\mathbf{n}_\perp \neq 0$ separately from the case $\mathbf{n}_\perp = 0$. Indeed the isolation of the PV motion can be performed by a Fourier version of the canonical transformation (26)-(31) *for the case* $\mathbf{n}_\perp \neq 0$. The case $\mathbf{n}_\perp = 0$ must be treated differently, however normal mode coordinates are still easy to find.

In accordance with these observations, we write

$$H_2(\boldsymbol{q},\boldsymbol{p}) = H_\perp(\boldsymbol{q}(\mathbf{n}),\boldsymbol{p}(\mathbf{n})) + H_V(\boldsymbol{q}(0,0,n_3),\boldsymbol{p}(0,0,n_3)) \tag{48}$$

where $H_\perp$ is the part of the Hamiltonian involving a sum over Fourier coefficients satisfying $\mathbf{n}_\perp \neq 0$, and H_V (V for "vertical") is the remaining part of the Hamiltonian with $\mathbf{n}_\perp = 0$.

A. The Case of Vanishing Horizontal Wavenumber ($\mathbf{n}_\perp = 0$)

For convenience we write

$$\begin{aligned} \boldsymbol{q}(0,0,n_3) &= \boldsymbol{q}(n_3), \qquad (49) \\ \boldsymbol{p}(0,0,n_3) &= \boldsymbol{p}(n_3). \qquad (50) \end{aligned}$$

Then from Eq. (46), we may write

$$\begin{aligned} H_V(\boldsymbol{q},\boldsymbol{p}) &= \frac{1}{2}\sum_{n_3}\Big[\boldsymbol{p}_\perp(n_3)\cdot\boldsymbol{p}_\perp(-n_3) + f(q_2(n_3)p_1(n_3) - q_1(n_3)p_2(n_3)) \\ &+ \frac{f^2}{4}\boldsymbol{q}_\perp(n_3)\cdot\boldsymbol{q}_\perp(-n_3) + p_3(n_3)p_3(-n_3) + \gamma_{n_3}^2 q_3(n_3)q_3(-n_3)\Big], \end{aligned} \tag{51}$$

where

$$\gamma_{n_3}^2 \equiv \kappa^2 n_3^2 c^2 + \frac{\alpha^2 c^2}{4}. \tag{52}$$

For these $\mathbf{n}_\perp = 0$ modes the motion in q_3, p_3 is already explicitly decoupled from $\boldsymbol{q}_\perp$, $\boldsymbol{p}_\perp$. It is clear that the evolution of the q_3, p_3 represents acoustic waves with allowed frequencies

$\omega_{n_3} = \pm\gamma_{n_3}$. A canonical transformation that diagonalizes the remainder of H_V is

$$a_{PV}(n_3) = \frac{1}{2}q_1(n_3) + \frac{1}{f}p_2(-n_3) \tag{53}$$

$$b_{PV}(n_3) = -\frac{f}{2}q_2(-n_3) + p_1(n_3) \tag{54}$$

$$a_I(n_3) = \frac{1}{2}q_2(-n_3) + \frac{1}{f}p_1(n_3) \tag{55}$$

$$b_I(n_3) = -\frac{f}{2}q_1(n_3) + p_2(-n_3), \tag{56}$$

and for uniformity of notation we will also set

$$a_A(n_3) = q_3(n_3) \tag{57}$$

$$b_A(n_3) = p_3(n_3). \tag{58}$$

The notation PV, I, and A indicates potential vorticity, internal wave, and acoustic modes respectively. It can be checked easily that the a's and the b's form conjugate pairs

$$\{a_i(n_3), b_j(m_3)\} = \delta_{ij}\delta_{n_3 m_3}, \tag{59}$$

with the remaining brackets vanishing. The indices i and j now run over PV, I, and A. In terms of these variables we have

$$\begin{aligned} H_V(\mathbf{a},\mathbf{b}) &= \frac{1}{2}\sum_{n_3}[b_I(n_3)b_I(-n_3) + f^2 a_I(n_3)a_I(-n_3) \\ &+ b_A(n_3)b_A(-n_3) + \gamma_{n_3}^2 a_A(n_3)a_A(-n_3)]. \end{aligned} \tag{60}$$

We see that the PV degrees of freedom are completely absent from H_V, and the allowed internal wave frequencies are simply $\pm f$. These results are in agreement with the fact that f is the minimum allowed linear internal wave frequency and is reached when $\mathbf{k}_\perp = \lambda\mathbf{n}_\perp = 0$. PV or geostrophic motions vanish when $\mathbf{n}_\perp = 0$. We now proceed to the more complicated case when $\mathbf{n}_\perp \neq 0$.

B. The Case of Nonvanishing Horizontal Wavenumber ($\mathbf{n}_\perp \neq 0$)

The part of the Hamiltonian involving a sum over the $\mathbf{n}_\perp \neq 0$ modes is

$$\begin{aligned} H_\perp(\boldsymbol{q},\boldsymbol{p}) &= \frac{1}{2}\sum_{\mathbf{n}}{}' \Big[\boldsymbol{p}(\mathbf{n}) \cdot \boldsymbol{p}(-\mathbf{n}) + f(q_2(\mathbf{n})p_1(\mathbf{n}) - q_1(\mathbf{n})p_2(\mathbf{n})) + \frac{f^2}{4}\boldsymbol{q}_\perp(\mathbf{n}) \cdot \boldsymbol{q}_\perp(-\mathbf{n}) \\ &+ c^2(\lambda \mathbf{n}_\perp \cdot \boldsymbol{q}_\perp(\mathbf{n}) + \kappa n_3 q_3(\mathbf{n}))(\lambda \mathbf{n}_\perp \cdot \boldsymbol{q}_\perp(-\mathbf{n}) + \kappa n_3 q_3(-\mathbf{n})) \\ &+ i\lambda(\frac{\alpha c^2}{2} - g)(\mathbf{n}_\perp \cdot \boldsymbol{q}_\perp(\mathbf{n}) q_3(-\mathbf{n}) - \mathbf{n}_\perp \cdot \boldsymbol{q}_\perp(-\mathbf{n}) q_3(\mathbf{n})) + \frac{\alpha^2 c^2}{4} q_3(\mathbf{n}) q_3(-\mathbf{n}) \Big]. \end{aligned} \tag{61}$$

The prime on the summation symbol indicates that the terms $\mathbf{n}_\perp = 0$ are omitted from the sum. A Fourier version of the canonical transformation in Eq.'s (26)-(31) will now isolate the PV motion from the internal wave and acoustic wave motions. This Fourier version of the transformation may be written

$$Q_1(\mathbf{n}) = i\lambda(n_1 q_2(\mathbf{n}) - n_2 q_1(\mathbf{n})), \tag{62}$$

$$Q_2(\mathbf{n}) = i\lambda(n_1 q_1(\mathbf{n}) + n_2 q_2(\mathbf{n})), \tag{63}$$

$$Q_3(\mathbf{n}) = q_3(\mathbf{n}), \tag{64}$$

$$P_1(\mathbf{n}) = \frac{-i}{\lambda n_\perp{}^2}\left[n_1 p_2(\mathbf{n}) - n_2 p_1(\mathbf{n}) + \frac{f}{2}(n_1 q_1(-\mathbf{n}) + n_2 q_2(-\mathbf{n})) \right], \tag{65}$$

$$P_2(\mathbf{n}) = \frac{-i}{\lambda n_\perp{}^2}\left[n_1 p_1(\mathbf{n}) + n_2 p_2(\mathbf{n}) + \frac{f}{2}(n_1 q_2(-\mathbf{n}) - n_2 q_1(-\mathbf{n})) \right], \tag{66}$$

$$P_3(\mathbf{n}) = p_3(\mathbf{n}). \tag{67}$$

From these formulae it is evident that the $\mathbf{n}_\perp = 0$ modes did require separate treatment. It is again simple to verify that the Q's and P's form canonically conjugate pairs

$$\{Q_i(\mathbf{n}), P_j(\mathbf{m})\} = \delta_{ij}\delta_{\mathbf{nm}}, \tag{68}$$

with the remaining brackets vanishing. In terms of these coordinates we have

$$H_\perp(\mathbf{Q},\mathbf{P}) = \frac{1}{2}\sum_{\mathbf{n}}{}' [\lambda^2 n_\perp{}^2 (P_1(\mathbf{n})P_1(-\mathbf{n}) + P_2(\mathbf{n})P_2(-\mathbf{n})) + P_3(\mathbf{n})P_3(-\mathbf{n})$$

$$
\begin{aligned}
&+ \frac{\beta_{n_\perp}^2}{\lambda^2 {n_\perp}^2} Q_2(\mathrm{n}) Q_2(-\mathrm{n}) + \gamma_{n_3}^2 Q_3(\mathrm{n}) Q_3(-\mathrm{n}) - 2f Q_2(\mathrm{n}) P_1(\mathrm{n}) \\
&+ c(\sigma_1 + i\sigma_2(n_3)) Q_3(\mathrm{n}) Q_2(-\mathrm{n}) + c(\sigma_1 - i\sigma_2(n_3)) Q_2(\mathrm{n}) Q_3(-\mathrm{n})], \qquad (69)
\end{aligned}
$$

where

$$
\gamma_{n_3}^2 \equiv \kappa^2 n_3^2 c^2 + \frac{\alpha^2 c^2}{4}, \quad \beta_{n_\perp}^2 \equiv f^2 + \lambda^2 {n_\perp}^2 c^2, \quad \sigma_1 \equiv \frac{\alpha c}{2} - \frac{g}{c}, \quad \sigma_2(n_3) \equiv \kappa n_3 c. \qquad (70)
$$

Once our intermediate canonical transformation given by Eq. (34) is accounted for, $H_\perp(\mathbf{Q}, \mathbf{P})$ given in Eq. (69) is the Fourier space version of $H_2(\mathbf{Q}, \mathbf{P})$ given in Eq. (32). We see that $Q_1(\mathrm{n})$ does not occur in $H_\perp(\mathbf{Q}, \mathbf{P})$, so the $P_1(\mathrm{n}, t)$ are conserved for each $\mathbf{n}$. Therefore Q_1 and P_1 describe the PV degrees of freedom. The term involving $Q_2(\mathrm{n}) P_1(\mathrm{n})$ in Eq. (69) appears to couple the PV motion to the internal wave and acoustic modes. The constancy of the $P_1(\mathrm{n})$ means that this term represents just a constant displacement of either the acoustic or internal wave modes. These displacements can be removed from the quadratic Hamiltonian by a noncanonical change of variables, but we choose to carry them along, see Arnol'd (1978). We now give one further canonical transformation that serves to explicitly decouple the internal wave and acoustic modes

$$
A_A(\mathrm{n}) = \frac{1}{\lambda n_\perp} \sqrt{\frac{\omega_+^2(\mathrm{n}) - \gamma_{n_3}^2}{\omega_+^2(\mathrm{n}) - \omega_-^2(\mathrm{n})}} \left(Q_2(\mathrm{n}) + \frac{\gamma_{n_3}^2 - \omega_-^2(\mathrm{n})}{c(\sigma_1 - i\sigma_2(n_3))} Q_3(\mathrm{n}) \right), \qquad (71)
$$

$$
A_I(\mathrm{n}) = \frac{1}{\lambda n_\perp} \sqrt{\frac{\gamma_{n_3}^2 - \omega_-^2(\mathrm{n})}{\omega_+^2(\mathrm{n}) - \omega_-^2(\mathrm{n})}} \left(Q_2(\mathrm{n}) - \frac{\omega_+^2(\mathrm{n}) - \gamma_{n_3}^2}{c(\sigma_1 - i\sigma_2(n_3))} Q_3(\mathrm{n}) \right), \qquad (72)
$$

$$
\begin{aligned}
B_A(\mathrm{n}) &= \frac{\lambda n_\perp}{\sqrt{(\omega_+^2(\mathrm{n}) - \omega_-^2(\mathrm{n}))(\omega_+^2(\mathrm{n}) - \gamma_{n_3}^2)}} \\
&\times ((\omega_+^2(\mathrm{n}) - \gamma_{n_3}^2) P_2(\mathrm{n}) + c(\sigma_1 - i\sigma_2(n_3)) P_3(\mathrm{n})), \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
B_I(\mathrm{n}) &= \frac{\lambda n_\perp}{\sqrt{(\omega_+^2(\mathrm{n}) - \omega_-^2(\mathrm{n}))(\gamma_{n_3}^2 - \omega_-^2(\mathrm{n}))}} \\
&\times ((\gamma_{n_3}^2 - \omega_-^2(\mathrm{n})) P_2(\mathrm{n}) - c(\sigma_1 - i\sigma_2(n_3)) P_3(\mathrm{n})), \qquad (74)
\end{aligned}
$$

where

$$\begin{aligned} 2\omega_{\pm}^2(\mathbf{n}) &= \beta_{n_\perp}^2 + \gamma_{n_3}^2 \pm [(\beta_{n_\perp}^2 + \gamma_{n_3}^2)^2 - 4(N^2\beta_{n_\perp}^2 + f^2(\gamma_{n_3}^2 - N^2))]^{1/2} \\ &= \beta_{n_\perp}^2 + \gamma_{n_3}^2 \pm \sqrt{(\beta_{n_\perp}^2 - \gamma_{n_3}^2)^2 + 4\lambda^2 c^2 {n_\perp}^2 |\sigma|^2}, \end{aligned} \tag{75}$$

and where $\sigma = \sigma_1 + i\sigma_2$. Note that $|\sigma|^2 = \gamma_{n_3}^2 - N^2$. We see from this that both $\omega_{\pm}^2$ are real and positive. Also we see that the quantities $\gamma_{n_3}^2 - \omega_{-}^2$ and $\omega_{+}^2 - \gamma_{n_3}^2$ are positive. All the square roots in the canonical transformation are therefore well defined. Again for uniformity of notation we also put

$$A_{PV}(\mathbf{n}) = Q_1(\mathbf{n}), \quad B_{PV}(\mathbf{n}) = P_1(\mathbf{n}). \tag{76}$$

This transformation may appear mysterious, however, it can be motivated by a standard normal mode analysis of the linear equations of motion for **Q** and **P**. $\omega_{\pm}^2$ are simply the solutions of the quadratic equation for ω^2 in the dispersion relation (39), where the wave numbers have been properly "quantized" through the imposition of periodic boundary conditions

$$\mathbf{k} = (\lambda n_1, \lambda n_2, \kappa n_3). \tag{77}$$

The A's and the B's form canonically conjugate pairs

$$\{A_i(\mathbf{n}), B_j(\mathbf{m})\} = \delta_{ij}\delta_{\mathbf{nm}}, \tag{78}$$

with the remaining brackets vanishing.

Note that as the speed of sound becomes infinite,

$$\omega_{+}^2 \rightarrow |\mathbf{k}|^2 c^2 = (\lambda^2 n_\perp^2 + \kappa^2 n_3^2)c^2, \tag{79}$$

and

$$\omega_{-}^2 \rightarrow \frac{k_\perp^2 N^2 + k_3^2 f^2}{k_\perp^2 + k_3^2}. \tag{80}$$

Now we express the $\mathbf{n}_\perp \neq 0$ part of the Hamiltonian, $H_\perp$ in terms of the normal mode *coordinates*

$$\begin{aligned} H_\perp(\mathbf{A},\mathbf{B}) &= \frac{1}{2}\sum_{\mathbf{n}}{}' \Big\{ \lambda^2 {n_\perp}^2 B_{PV}(\mathbf{n})B_{PV}(-\mathbf{n}) + B_A(\mathbf{n})B_A(-\mathbf{n}) + B_I(\mathbf{n})B_I(-\mathbf{n}) \\ &+ \omega_+^2(\mathbf{n})A_A(\mathbf{n})A_A(-\mathbf{n}) + \omega_-^2(\mathbf{n})A_I(\mathbf{n})A_I(-\mathbf{n}) \\ &- 2f\lambda n_\perp B_{PV}(\mathbf{n}) \left[\sqrt{\frac{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} A_A(\mathbf{n}) + \sqrt{\frac{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} A_I(\mathbf{n})\right]\Big\}. \end{aligned} \tag{81}$$

From Hamilton's equations

$$\frac{dA_i(\mathbf{n})}{dt} = \frac{\partial H_\perp(\mathbf{A},\mathbf{B})}{\partial B_i(\mathbf{n})}, \quad \frac{dB_i(\mathbf{n})}{dt} = -\frac{\partial H_\perp(\mathbf{A},\mathbf{B})}{\partial A_i(\mathbf{n})},$$

we can now deduce the normal mode equations for the evolution of either the momenta $\mathbf{B}(\mathbf{n},t)$ or the coordinates $\mathbf{A}(\mathbf{n},t)$,

$$\begin{aligned} \dot{A}_{PV}(\mathbf{n},t) &= \lambda^2 {n_\perp}^2 B_{PV}^*(\mathbf{n},t) - f\lambda n_\perp \left[\sqrt{\frac{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} A_A(\mathbf{n},t)\right. \\ &+ \left.\sqrt{\frac{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} A_I(\mathbf{n},t)\right], \\ \dot{B}_{PV}(\mathbf{n},t) &= 0, \\ \dot{A}_A(\mathbf{n},t) &= B_A^*(\mathbf{n},t), \\ \dot{B}_A^*(\mathbf{n},t) &= -\omega_+^2(\mathbf{n})A_A(\mathbf{n},t) + f\lambda n_\perp \sqrt{\frac{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} B_{PV}(\mathbf{n},t), \\ \dot{A}_I &= B_I^*(\mathbf{n},t), \\ \dot{B}_I^*(\mathbf{n},t) &= -\omega_-^2(\mathbf{n})A_I(\mathbf{n},t) + f\lambda n_\perp \sqrt{\frac{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}} B_{PV}(\mathbf{n},t). \end{aligned} \tag{82}$$

So we do indeed have normal mode coordinates.

VI. Summary; Future Work

The work in this paper has concentrated solely on the identification of the linear normal modes for the evolution of an inviscid, stratified fluid—all done in Lagrangian formulation. We have worked with a compressible fluid, so the three kinds of linear modes are an acoustic mode, internal wave modes, and a low frequency potential vorticity carrying mode which satisfies the geostrophic balance. The Hamiltonian formulation of the problem in Lagrangian fluid variables provides both a fully diagonalized Hamiltonian for the linear problem and a framework for the interaction of these linear modes in a fully nonlinear, fully Hamiltonian fashion.

To provide a summary in one place of the developments and various canonical transformations developed in this paper, we collect the results here. First of all we have the Lagrangian coordinates $\mathbf{Y}(\mathbf{r},t) = \mathbf{r} + \mathbf{X}(\mathbf{r},t)$ and canonical momenta $\boldsymbol{\Pi}(\mathbf{r},t) = \hat{\mathbf{z}} \times \mathbf{r}\, f/2 + \boldsymbol{\pi}(\mathbf{r},t)$ which satisfy the canonical Poisson bracket relation $\{X_i(\mathbf{r}), \pi_j(\mathbf{r}')\} = \delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}')$ with all other brackets vanishing. The Hamiltonian in these canonical coordinates takes the form

$$H(\mathbf{X},\boldsymbol{\pi}) = \int d^3r \left[\frac{|\boldsymbol{\pi} - \rho_0 \mathbf{R}(\mathbf{X})|^2}{2\rho_0} + \rho_0 g X_3 + \rho_0 \epsilon(\frac{\rho_0}{1+\xi})\right], \tag{83}$$

where the quantity ξ is

$$\xi = 1 + \nabla\cdot\mathbf{X} + \frac{\partial(X_1,X_2)}{\partial(r_1,r_2)} + \frac{\partial(X_1,X_3)}{\partial(r_1,r_3)} + \frac{\partial(X_2,X_3)}{\partial(r_2,r_3)} + \frac{\partial(\mathbf{X})}{\partial(\mathbf{r})}. \tag{84}$$

The expansion of this Hamiltonian to second order in powers of $\mathbf{X}(\mathbf{r},t)$ was found to be

$$H_2(\mathbf{X},\boldsymbol{\pi}) = \int d^3r \left[\frac{|\boldsymbol{\pi}|^2}{2\rho_0} + \frac{f}{2}[X_2\pi_1 - X_1\pi_2] + \frac{\rho_0 f^2}{8}X_\perp^2 + \frac{\rho_0 c^2}{2}(\nabla\cdot\mathbf{X})^2 - \rho_0 g X_3 \nabla_\perp \cdot X_\perp\right], \tag{85}$$

using $\partial p_0(r_3)/\partial r_3 = -g\rho_0(r_3)$ and $p_0 = \rho_0^2 \epsilon_\rho(\rho_0)$. This is the quadratic Hamiltonian we diagonalized in the body of this paper. To do so we performed various canonical transformations from the phase space coordinates $\mathbf{X}$ and $\boldsymbol{\pi}$ to others which evolve as simple harmonic oscillators. In a moment we will make the complete connection between these normal mode coordinates and the $\mathbf{X}$ and $\boldsymbol{\pi}$ coordinates.

We may use the normal canonical coordinates in the full Hamiltonian $H(\mathbf{X}, \boldsymbol{\pi})$ written above and, with Hamilton's equations, have a set of fluid evolution equations which are not amenable to analytic solution but can be addressed numerically with the advantage over the formulation in $\mathbf{X}$, $\boldsymbol{\pi}$ form that we know exactly the linear interpretation of each of the modes: acoustic (A), internal wave (I), and potential vorticity (PV) as they enter the Hamiltonian. The PV modes enter in a nice way in the quadratic Hamiltonian: only the PV momenta, called $B_{PV}(\mathbf{r}, t)$, are present. Their canonical conjugates, called $A_{PV}(\mathbf{r}, t)$, are absent, so the B_{PV} are constant in time. The $A_{PV}(\mathbf{r}, t)$ are present in the potential energy term $\rho_0 g X_3 + \rho_0 \epsilon(\frac{\rho_0}{1+\xi})$. This means that the $B_{PV}(\mathbf{r}, t)$ will not be conserved in the nonlinear interactions. It is interesting to note that the conserved momentum $B_{PV}(\mathbf{r}, t)$ *is* found in the potential energy, so the dynamics of regions of the fluid with different potential vorticity will generally be different. This would seem consistent with some of the observations of Sommeria et al. (1989).

One may regard this as a problem, but it is not a serious one. This is for two reasons. We know from general considerations, Abarbanel and Holm (1987), the exact form of the conserved potential vorticity $\Omega(\mathbf{r}, t)$

$$\rho_0(r_3)\Omega(\mathbf{r}, t) = \sum_{n=1}^{3} \left\{ \frac{\partial Y_n}{\partial r_1} \frac{\partial \Pi_n}{\partial r_2} - \frac{\partial Y_n}{\partial r_2} \frac{\partial \Pi_n}{\partial r_1} \right\}, \tag{86}$$

and this is proportional to $B_{PV}(\mathbf{r}, t)$ in linear order. The full $\Omega(\mathbf{r}, t)$ is quadratic in the coordinates and can be easily tracked throughout any investigation of the nonlinear interactions. In other words, the exact potential vorticity can be identified in the interaction and in the solution of the equations of motion.

There is another attractive possibility which we will investigate in our subsequent papers on this subject. The full potential vorticity is only a quadratic polynomial in canonical coordinates It may be possible to make a canonical transformation in which $\Omega(\mathbf{r})$ becomes a new "momentum" with its conjugate "coordinate" absent from the Hamiltonian. Then the conservation law will be evident but the presence of $\Omega(\mathbf{r})$ in the interaction potential energy will still provide a driving of internal wave and acoustic motions by PV states.

One may use this observation to set up an interesting problem where working to some order in the nonlinearity one specifies the PV modes at $t = 0$ knowing they do not evolve (on an f-plane). They then act as time independent driving terms for the other modes. If one establishes a mesoscale flow at an initial time, its energy will transfer to internal wave motions as the system evolves. The rate of transfer can be compared to the rates observed by Brown and Evans (1981).

The decoupling of the acoustic modes when $c^2 \to \infty$ is certainly expected on physical grounds, but we have not exhibited this explicitly. While we will deal with this in our future work, we have two observations which may be useful in this regard. First, there actually may not be a smooth transition in a formal sense to infinite sound speed since the Hamiltonian formulation of that problem *ab initio* requires introducing the pressure as a Lagrange multiplier which serves to enforce the constraint that the Jacobian $\partial(\mathbf{Y}(\mathbf{r},t))/\partial(\mathbf{r})$ always equal unity. This departs from the straight line Hamiltonian path we have developed here and requires complications to introduce this constraint in Poisson brackets and other necessary formal apparatus, Abarbanel et al. (1986). If one has to avoid the limit in the Hamiltonian itself, then noting that the ω_+ modes are much more rapidly evolving than either the internal wave or PV modes when c is large, we may eliminate the acoustic modes by either some form of averaging or further, approximate canonical transformations to the "adiabatic invariant" associated with the rapid motion.

We complete the cycles of canonical transformation given in this paper by recording the full expression of the conjugate coordinates $\mathbf{X}(\mathbf{r},t)$ and $\boldsymbol{\pi}(\mathbf{r}.t)$ in terms of the normal modes. First we recall that the $\mathbf{X}$ and $\boldsymbol{\pi}$ have Fourier expansions

$$\begin{aligned} X_i(\mathbf{r},t) &= \frac{1}{\sqrt{\rho_0 DL^2}}\{\sum_{n_3;\mathbf{n}_\perp=0} q_i(n_3,t)\exp(i\kappa n_3 r_3) \\ &+ \sum_{\mathbf{n};\mathbf{n}_\perp\neq 0} q_i(\mathbf{n},t)\exp(i(\lambda\mathbf{n}_\perp\cdot\mathbf{r}_\perp+\kappa n_3 r_3))\}, \end{aligned} \tag{87}$$

$$\begin{aligned} \pi_i(\mathbf{r},t) &= \sqrt{\frac{\rho_0}{DL^2}}\{\sum_{n_3;\mathbf{n}_\perp=0} p_i(n_3,t)\exp(-i\kappa n_3 r_3) \\ &+ \sum_{\mathbf{n};\mathbf{n}_\perp\neq 0} p_i(\mathbf{n},t)\exp(-i(\lambda\mathbf{n}_\perp\cdot\mathbf{r}_\perp+\kappa n_3 r_3))\}. \end{aligned} \tag{88}$$

In terms of the conjugate variables $A_{PV}(\mathbf{n})$, $B_{PV}(\mathbf{n})$, $A_I(\mathbf{n})$, $B_I(\mathbf{n})$, $A_A(\mathbf{n})$, and $B_A(\mathbf{n})$ we have the total quadratic Hamiltonian

$$\begin{aligned} H(\mathbf{A},\mathbf{B}) &= \frac{1}{2}\sum_{\mathbf{n}}\Big\{\lambda^2 n_\perp{}^2 B_{PV}(\mathbf{n})B_{PV}(-\mathbf{n}) + B_A(\mathbf{n})B_A(-\mathbf{n}) + B_I(\mathbf{n})B_I(-\mathbf{n}) \\ &+ \omega_+^2(\mathbf{n})A_A(\mathbf{n})A_A(-\mathbf{n}) + \omega_-^2(\mathbf{n})A_I(\mathbf{n})A_I(-\mathbf{n}) \\ &- 2f\lambda n_\perp B_{PV}(\mathbf{n})\left[\sqrt{\frac{\omega_+^2(\mathbf{n})-\gamma_{n_3}^2}{\omega_+^2(\mathbf{n})-\omega_-^2(\mathbf{n})}}A_A(\mathbf{n}) + \sqrt{\frac{\gamma_{n_3}^2-\omega_-^2(\mathbf{n})}{\omega_+^2(\mathbf{n})-\omega_-^2(\mathbf{n})}}A_I(\mathbf{n})\right]\Big\} \end{aligned} \tag{89}$$

where we have combined the two forms for $\mathbf{n}_\perp = 0$ and $\mathbf{n}_\perp \neq 0$, noting that at $\mathbf{n}_\perp = 0$, $\omega_+^2 = \gamma_{n_3}^2$, and $\omega_-^2 = f^2$. Also we have used the notation that $A_i(\mathbf{n}_\perp = 0, n_3) = a_i(n_3)$ and $B_i(\mathbf{n}_\perp = 0, n_3) = b_i(n_3)$ to conform to our earlier symbols. The equations of motion at linear order are those given in Eq.(82).

Finally we record the connections between the $q_i(\mathbf{n})$ and $p_i(\mathbf{n})$ and the normal mode coordinates. First for $\mathbf{n}_\perp = 0$ we have

$$q_1(n_3) = a_{PV}(n_3) - \frac{b_I(n_3)}{f}, \tag{90}$$

$$q_2(n_3) = a_I^*(n_3) - \frac{b_{PV}^*(n_3)}{f}, \tag{91}$$

$$p_1(n_3) = \frac{1}{2}[f a_I(n_3) + b_{PV}(n_3)], \tag{92}$$

$$p_2(n_3) = \frac{1}{2}[f a_{PV}^*(n_3) + b_I^*(n_3)]. \tag{93}$$

For the more complex case when $\mathbf{n}_\perp \neq 0$ we have the following

$$q_1(\mathbf{n}) = \frac{-i}{\lambda n_\perp}(n_1 Q_2(\mathbf{n}) - n_2 Q_1(\mathbf{n})), \tag{94}$$

$$q_2(\mathbf{n}) = \frac{-i}{\lambda n_\perp}(n_1 Q_1(\mathbf{n}) + n_2 Q_2(\mathbf{n})), \tag{95}$$

$$q_3(\mathbf{n}) = Q_3(\mathbf{n}), \tag{96}$$

where

$$Q_1(\mathbf{n}) = A_{PV}(\mathbf{n}), \tag{97}$$

$$Q_2(\mathbf{n}) = \frac{\lambda n_\perp}{\sqrt{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}}\left(\sqrt{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}A_A(\mathbf{n}) + \sqrt{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}A_I\right), \tag{98}$$

$$Q_3(\mathbf{n}) = \frac{\lambda n_\perp c(\sigma_1 - i\sigma_2(n_3))}{\sqrt{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}}\left(\frac{A_A(\mathbf{n})}{\sqrt{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}} - \frac{A_I(\mathbf{n})}{\sqrt{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}}\right). \tag{99}$$

Also we have

$$p_1(\mathbf{n}) = i\lambda(n_1 P_2(\mathbf{n}) - n_2 P_1(\mathbf{n})) - \frac{if}{2\lambda n_\perp}[n_1 Q_1^*(\mathbf{n}) - n_2 Q_2^*(\mathbf{n})], \tag{100}$$

$$p_2(\mathbf{n}) = i\lambda(n_1 P_1(\mathbf{n}) + n_2 P_2(\mathbf{n})) - \frac{if}{2\lambda n_\perp}[n_1 Q_2^*(\mathbf{n}) + n_2 Q_1^*(\mathbf{n})], \tag{101}$$

$$p_3(\mathbf{n}) = P_3(\mathbf{n}), \tag{102}$$

and finally

$$P_1(\mathbf{n}) = B_{PV}(\mathbf{n}), \tag{103}$$

$$P_2(\mathbf{n}) = \frac{1}{\lambda n_\perp\sqrt{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}}\left(\sqrt{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}B_A(\mathbf{n}) + \sqrt{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}B_I(\mathbf{n})\right), \tag{104}$$

$$P_3(\mathbf{n}) = \frac{1}{\lambda n_\perp c(\sigma_1 - i\sigma_2(n_3))}\sqrt{\frac{(\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2)(\gamma_{n_3}^2 - \omega_-^2(\mathbf{n}))}{\omega_+^2(\mathbf{n}) - \omega_-^2(\mathbf{n})}}$$
$$\times \left(\sqrt{\gamma_{n_3}^2 - \omega_-^2(\mathbf{n})}B_A(\mathbf{n}) - \sqrt{\omega_+^2(\mathbf{n}) - \gamma_{n_3}^2}B_I(\mathbf{n})\right). \qquad (105)$$

These connections summarize the set of canonical transformations we have developed to transform the Lagrangian fluid variables $\mathbf{X}(\mathbf{r}, t)$ and $\pi(\mathbf{r}, t)$ to normal modes of the linearized problem of stratified flow. Using these expressions in the full nonlinear Hamiltonian Eq.(83) will allow the study of the interaction among the PV, I, and A modes to all orders while keeping the physical identification we have established. An analysis of several of the issues and suggestions made here will be presented in our subsequent work.

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